Lesson 1

Topic: Number and place value

Establishing index notation

Lesson concepts

- **Real numbers** — Index notation
- **Equivalence** — Simplifying expressions
- **Representations** — Algebraic conventions
- **Representation** — Generalisations

Lesson notes

Students will:

- review and extend the commutative law
- review and extend the associative law
- express numbers using index notation
- convert algebraic expressions using index notation.

Lesson answers

Exploration 1

Puzzle 1: A lily pad grows so that each day it doubles its size (area). On the 20th day of its life, it completely covers the pond. On what day of its life was the pond half covered?

Because it doubles in area every day, it would have covered half the area the day before, which is day 19.

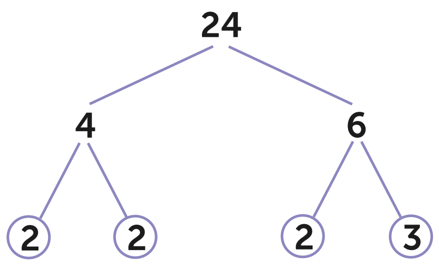
Puzzle 2: If your parents offered you two different options for receiving your pocket money, which of the following would you choose? Option A: $200 per week Option B: One cent paid on the first day of the month, two cents paid on the second day, 4 cents paid on the third day and so on, with the amount received each day doubling until the end of the month.

Option B is the better option. This demonstrates how quickly things grow exponential (such as when the amount doubles every day). Consider February with 28 days. The amount you would earn would be $1+2+4+8+16+.....2^{27}$ cents. This is in fact $2^{28}-1$ = $2,684,354.55c

1. a. **No** $7 - 5 = 2$, which is not the same as $5 - 7$
   b. **No** $14 \div 7 = 2$, which is not the same as $7 \div 14$
   c. **No** $k - p$ will not always be equal to $p - k$
   d. **No** $r \div f$ will not always be equal to $f \div r$
2. a. **No** $(7 - 5) - 3 = -1$, which is not the same as $7 - (5 - 3)$
   b. **No** $(14 \div 7) \div 2 = 1$, which is not the same as $14 \div (7 \div 2)$
   c. **No** $(k - p) - g$ will not always be equal to $k - (p - g)$
   d. **No** $(r \div f) \div h$ will not always be equal to $r \div (f \div h)$

3. Students watch the **Video — Constructing a factor tree.**

4. There are many different ways to construct this factor tree. In this example we've chosen the factors of 4 and 6 to start with. The associative law says we could have started with 2 and 12 or 3 and 8. The commutative law says we could even have used 6 and 4, 12 and 2 or 8 and 3. As long as we end up with three factors of 2 and one factor of 3.



$2 \times 2 \times 2 \times 3$	$2^3 \times 3$	4×6	8×3	2×12	$2 \times 3 \times 4$
$2 \times 4 \times 3$	$3 \times 4 \times 2$	$3 \times 2 \times 4$	$4 \times 3 \times 2$	$4 \times 2 \times 3$	$2 \times 2 \times 3 \times 2$
$2 \times 3 \times 2 \times 2$	$3 \times 2 \times 2 \times 2$	6×4	3×8	12×2	

5. a. 4^6
   b. $3^2 \times 7^2$
   c. $6^3 \times 2^3$
6. a. i. p^4 (this first one was done for you)
      ii. k^6
      iii. $m^2 \times h^2$
      iv. $f^3 \times g^4$
   b. i. $f \times f \times f \times f \times f \times f \times f \times f$
      ii. $r \times r \times r \times r \times t \times t$
      iii. $3 \times k \times k \times j \times j$
      iv. $7 \times a \times y \times y \times y \times y \times y \times y$

Lesson 2

Topic: Number and place value

Establishing the multiplication index law on numbers

Lesson concepts

- **Real numbers** — Index notation
- **Equivalence** — Simplifying expressions
- **Representations** — Algebraic conventions
- **Representation** — Generalisations

Lesson notes

Students will:

- review writing algebraic index expressions in expanded form
- develop the multiplication index law with whole number bases
- generalise the multiplication index law.

Lesson answers

Exploration 2
WODB

$27x^2$	$3x^2$
$45x^2$	$9x^3$

There are many reasons why one may not belong. The main thing is that the description does not fit any of the other three.

Here is one possible set.

$27x^2$ because it is the only term with a coefficient that is a perfect cube

$3x^2$ because it is the only term with a coefficient that is a prime number

$45x^2$ because it is the only term with a coefficient where the tens digit and unit digits are consecutive positive numbers.

$9x^3$ because it is the other three terms are like terms (kx^2) and it not ($9x^3$)

1.

Index or factor form	Expanded form
a. 100,000 = 10^5	$10 \times 10 \times 10 \times 10 \times 10$
b. 16 = 2^4	$2 \times 2 \times 2 \times 2$
c. h^7	$h \times h \times h \times h \times h \times h \times h$
d. $5w^2$	$5 \times w \times w$
e. $8k^2p^3$	$k \times k \times 2 \times p \times p \times p \times 4$
f. $5r^3w^2$	$5 \times r \times r \times r \times w \times w$

2. a.

Number of rounds	1	2	3	4	5	6	7
Emails sent	6	36	216	1 296	7 776	46 656	279 936
Index form	6^1	6^2	6^3	6^4	6^5	6^6	6^7

b. 6^7 = 279 9366

c. 6^{12} = 2 176 782 336

d. The assumption is that each of the six people in round one sent it to 6 others. Someone may have realised it was a scam and deleted it. We also assume that they are not forwarding the email to someone who has already received the email.

e. The pattern would have the number 20 as the base and the number of rounds as the index. For example, 20^r

3. $8 \times 8 \times 8 \times 8 \times 3 = 8^4 \times 3$ OR 3×8^4

4. Mice is $7 \times 7 \times 7 = 7^3 = 343$

Corn = $7 \times 7 \times 7 \times 7 = 7^4$ = 2 401

Grain = $7 \times 7 \times 7 \times 7 \times 7 = 7^5$ = 16 807

The author miscalculated the 7 to the power of 4. It should be 2 401 and not 2 301.

5. a. $2^4 \times 2^2 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$
   b. $4^5 \times 4^3 = 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^8$
   c. $7^4 \times 7^1 = 7 \times 7 \times 7 \times 7 \times 7 = 7^5$
   d. $10^3 \times 10^2 = 10 \times 10 \times 10 \times 10 \times 10 = 10^5$
   e. $a^2 \times a^1 = a \times a \times a = a^3$
   f. $c^3 \times c^4 = c \times c \times c \times c \times c \times c \times c = c^7$

6. Sample response: I can see that when I have the same base, and multiply them, all I have to do is add the indices together. Provide the rule am × an = am+n, if requested. The rule works because the index is a counter counting the number of repeated factors. When you multiply two powers with the same repeating factors (same base), you can work out the total number of repeated factors by adding the counters (indices) together.

7. a. $3^4 \times 3^6 = 3^{10}$
   b. $7^9 \times 7^4 = 7^{13}$
   c. $8^6 \times 8^2 = 8^8$
   d. $k^8 \times k^4 = k^{12}$
   e. $w^3 \times w^5 = w^8$

   Answer check

   (3x3x3x3) x (3x3x3x3x3x3)

   =3x3x3x3x3x3x3x3x3x3

   $=3^{10}$

Lesson 3

Topic: Number and place value

Establishing the division index law on numbers

Lesson concepts

- **Real numbers** — Index notation
- **Equivalence** — Simplifying expressions
- **Representations** — Algebraic conventions
- **Representation** — Generalisations

Lesson notes

Students will:

- review the multiplication index law on numbers
- develop the division index law with whole number bases
- generalise the division index law
- use the division index law to develop the zero index law.

Lesson answers

Exploration 3

Acceptable groupings can vary as long as students can justify their choice.

Some possible groupings are shown below.

2^3	2x2x2	8	
2^4	2x2x2x2	16	4x4
$^-1^3$	$^-1$		
3^2	3x3		
1^5	1x1x1x1x1		
$2^3 \text{x} 2^4$	2x2x2x2x2x2x2	128	
$(^-1)^5$	$^-1$	$^-1\text{x}^-1\text{x}^-1\text{x}^-1\text{x}^-1$	
$^-2^4$	$^-16$		

1. a. (3 × 3) × (3 × 3 × 3 × 3 × 3)

   b. 3^7

   c. The index shows there are 2 factors of 3 multiplied by another 5 factors of 3 so the product will have 8 (=3+5) factors of 3. As the powers have the same base, adding the indices (counters of the repeated factors) finds the index for the power of 3 in the answer.

2. Students watch the **Video — Simplifying fractions.**

3.

   a. $\frac{3^5}{3^2} = \frac{3 \times 3 \times 3 \times 3 \times 3}{3 \times 3} = 3^3$

   b. $\frac{8^4}{8^2} = \frac{8 \times 8 \times 8 \times 8}{8 \times 8} = 8^2$

   c. $\frac{4^5}{4^5} = \frac{4 \times 4 \times 4 \times 4 \times 4}{4 \times 4 \times 4 \times 4 \times 4 \times 4} = \frac{1}{4}$

   d. $\frac{x^6}{x^3} = \frac{x \times x \times x \times x \times x \times x}{x \times x \times x} = x^3$

   e. $\frac{f^5}{f^4} = \frac{f \times f \times f \times f \times f}{f \times f \times f \times f} = f$

   f. $\frac{x^7}{x^2} = \frac{x \times x \times x \times x \times x \times x \times x}{x \times x} = x^5$

   g. $\frac{b^4}{b^1} = \frac{b \times b \times b \times b}{b} = b^3$

4. Sample response: I can see that when I have the same base, and divide them, all I have to do is subtract the indices. Provide the rule $a^m \div a^n = a^{m-n}$, if requested. The numerator and denominator have the same base, so the repeated factors are common. To simplify the fraction you can cancel the common factors. Because you are canceling factors, the index (counter of factors) of the numerator will be reduced by the number of factors canceled out, so you subtract the indices.

5. a. 5^1 or 5

   b. 12^6

   c. $\frac{4^5}{7^3}$

   d. $\frac{5^4}{5^3} = \frac{5 \times 5 \times 5 \times 5}{5 \times 5 \text{ x } 5} = 5$

   e. 5c. is different because there are powers of different bases but there are still common factors of 7 that cancel and common factors of 4 that will cancel.

   (7x7x7x 4x4x4x4x4x4x4)/(7x7x7x4x4)

   All the factors of 7 cancel and two factors of 4 cancel, leaving only five factors of 4.

6. a. $4^5 \div 4^5 = 4^{5-5} = 4^0 = 1$

   b. $8^3 \div 8^3 = 8^{3-3} = 8^0 = 1$

   c. $12^6 \div 12^6 = 12^{6-6} = 12^0 = 1$

   d. Sample response. 97 ÷ 97 = 97-7 = 90 = 1

7. If a=0 then $0^5 \div 0^5$ would be indeterminant. Try it on your calculator. You receive an error message. You can not divide by zero. Try 2÷0 on your calculator. 'How many zeros in 2?' cannot be determined.

Lesson 4

Topic: Number and place value

Establishing the 'power of a power' index law on numbers

Lesson concepts

- **Real numbers** — Index notation
- **Equivalence** — Simplifying expressions
- **Representations** — Algebraic conventions
- **Representation** — Generalisations

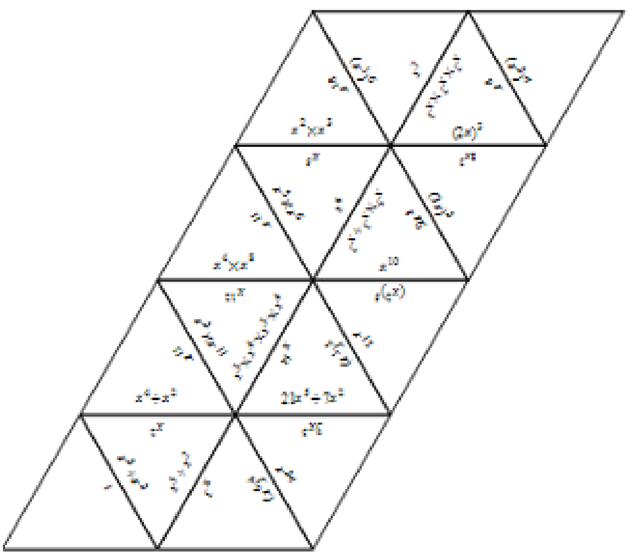
Lesson notes

Students will:

- review writing index expressions in expanded form
- develop the 'power of a power' law with whole number bases and positive indices
- generalise the 'power of a power' index law.

Lesson answers

Exploration 4



1.

Expanded form	Index form
$3 \times 3 \times 5 \times 5 \times 5$	$3^2 \times 5^3$
$p \times w \times w \times p \times w \times w$	$p^2\ w^4$

2. a. Sample response: -32 and -32

Expand $(-2)^5$	Expand -2^5
$(-2)^5$ $= -2 \times -2 \times -2 \times -2 \times -2$ $= +4 \times +4 \times -2$ $= +16 \times -2$ $= -32$	-2^5 $= -2 \times 2 \times 2 \times 2 \times 2$ $= -4 \times +4 \times 2$ $= -16 \times 2$ $= -32$

3. a. $(4^3)^5 = 4^3 \times 4^3 \times 4^3 \times 4^3 \times 4^3 = 4^{15}$

b. $(5^7)^3 = 5^7 \times 5^7 \times 5^7 = 5^{21}$

c. $(10^2)^5 = 10^2 \times 10^2 \times 10^2 \times 10^2 \times 10^2 = 10^{10}$

d. $3 \times (2^5)^6 = 3 \times 2^5 \times 2^5 \times 2^5 \times 2^5 \times 2^5 \times 2^5 = 3 \times 2^{30}$

e. $4 \times (9^2)^3 = 4 \times 9^2 \times 9^2 \times 9^2 = 4 \times 9^6$

f. $(3^4 \times 2^5)^2 = 3^4 \times 2^5 \times 3^4 \times 2^5 = 3^8 \times 2^{10}$

4. Sample response: When there is a power of a power, you multiply the powers.

5. a. a^{18}

b. m^{36}

c. $3k^4 \times 3k^4 = 9k^8$

Lesson 5-6

Topic: Number and place value

Solving problems using indices

Lesson concepts

- **Real numbers** — Index notation
- **Equivalence** — Simplifying expressions
- **Representations** — Algebraic conventions
- **Representation** — Generalisations

Lesson notes

Students will:

- revise writing numbers in different forms
- consolidate index laws on numbers
- generate and interpret index patterns in life-related problems.

Lesson answers

Exploration 5

See solution to Exploration 1

1.

2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
1	2	3	4	16	32	64	128	256	512	1 024

2. Students watch the **Video — Indices in various forms.**

3.

Index form	Power of a power	Power of primes
$16^3 = 4096$	**$(2^4)^3$**	**$2^{12} = 4096$**
8^2	$(2^3)^2$	2^6
128^3	$(2^7)^3$	2^{21}
4^3	$(2^2)^3$	2^6
4^7	$(2^2)^7$	2^{14}
32^2	$(2^5)^2$	2^{10}

4. Sheet 1 — Index laws on whole numbers (see page 14 for answers).

5.

	Parents	Grandparents	Greatgrand-parents	Great-great grandparents					
Number of generations	1	2	3	4	5	10	15	20	n
Number of ancestors	2	4	8	16	32	1 024	32 768	1 048 576	
	2^1	2^2	2^3	2^4	2^5	2^{10}	2^{15}	2^{20}	2^n

6. a. 2^{12} and 4096

b. Yes because 256 is 2^8

c. No because you cannot express 4 000 as a power of 2.

7. a. There are approx. 250 years from 1764 until now, but this is not a multiple of 20. If I consider 240 years (which is 12 lots of 20 years) then there are 12 generations between 1764 and now.

b. Both Joseph and Melissa have 2^{12} ancestors.

c. The assumption was that there was a new generation every 20 years. This would not be the case in modern times. If there was a generation every 25 years there would only be 10 generations in 250 years and so 2^{10} ancestors, which is a quarter of the previous answer.

8. a. $32^3 \div 16^2 = (2^5)^3 \div (2^4)^3 = 2^{15} \div 2^{12} = 2^{(15-12)} = 2^3$

b. $8^6 \div 4^5$

$= (2^3)^6 \div (2^2)^5$

$= 2^{18} \div 2^{10}$

$= 2^{18-10}$

$= 2^8$

c. $16^3 \div 8^4$

$= (2^4)^3 \div (2^3)^4$

$= 2^{12} \div 2^{12}$

$= 2^{12-12}$

$= 2^0$

$= 1$

d. $9^4 \times 243^2 =$

$=(3^2)^4 \times (3^5)^2$

$=3^8 \times 3^{10}$

$=3^{8+10}$

$= 3^{18}$

e. $(a^6)^3 \div (a^3)^5 =$

$= a^{6\times3} \div a^{3\times5}$

$= a^{18} \div a^{15}$

$= a^{18-15}$

$= a^3$

f. $(c^4)^2 \times (c^3)^4 =$

$= c^{4\times2} \times c^{3\times4}$

$= c^8 \times c^{12}$

$= c^{8+12}$

$= c^{20}$

Exploration6

https://nrich.maths.org/807

(2,4,6), (10,12,2) & (2,6,8) are all examples of triples of whole numbers (a,b,c) such that $a^2 + b^2 + c^2$ is a multiple of 4:

$2^2 + 4^2 + 6^2 = 56$
$10^2 + 12^2 + 2^2 = 248$
$2^2 + 6^2 + 8^2 = 104$

To find out if a, b and c must all be even, let's first consider what happens when all three numbers are even:
If any number is multiplied by 2, it is sure to be even.

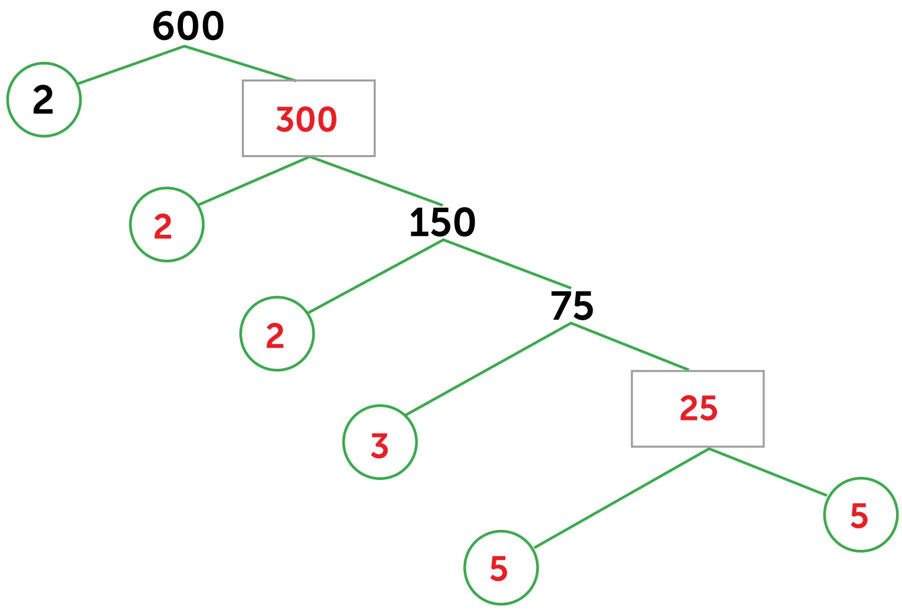
So let $a=2x$ $b=2y$ and $c=2z$
$(2x)^2 + (2y)^2 + (2z)^2 =$
$4x^2 + 4y^2 + 4z^2 =$
$4(x^2 + y^2 + z^2)$ which is a multiple of 4.

Students are unlikely to use an algebraic proof but a compelling written argument justification is perfectly acceptable.

Index laws on whole numbers

Note: Challenge questions are marked with an asterisk (*).

1. Fill in this factor tree for the number 600.



As a product of its prime factors, **600 =** 2 × 2 × 2 × 3 × 5 × 5

Using index notation, **600 =** 2^3 × 3 × 5^2

2. Fill in the following table to show all the powers of 3 up to 35.

3^0	3^1	3^2	3^3	3^4	3^5
1	3	9	27	81	243

3. Rewrite each expression in index notation.

a. $5 \times 5 \times 5 \times 5 \times 5 \times 5$

5^6

*c. $w \times w \times w \times w \times w \times w$

w^6

b. $6 \times 6 \times 6 \times 6 \times 8 \times 8 \times 8$

$6^4 \times 8^3$

*d. $y \times y \times y \times k \times k \times k \times k \times y \times y$

y^5k^4

4. Find the missing value in the following equations:

a. $4^3 \times 4^5 = 4^?$

$4^3 \times 4^5 = 4^8$

*e. $d^9 \times d^7 = d^?$

$d^9 \times d^7 = d^{16}$

b. $7 \times 7 \times 7^6 = 7^?$

$7 \times 7 \times 7^6 = 7^8$

*f. $n^? \times n^5 = n^{20}$

$n^{15} \times n^5 = n^{20}$

c. $9^? \times 9^3 = 9^6$

$9^3 \times 9^3 = 9^6$

*g. $p^7 \times u^? \times p^? \times u^5 = p^{15}\, u^{12}$

$p^7 \times u^7 \times p^8 \times u^5 = p^{15}\, u^{12}$

d. $12^2 \times 12^? = 12^{12}$

$12^2 \times 12^{10} = 12^{12}$

5. Find the missing value in the following equations:

a. $8^9 \div 8^6 = 8^?$

$8^9 \div 8^6 = 8^3$

b. $\frac{3^{10}}{3^5} = 3^?$

$\frac{3^{10}}{3^5} = 3^5$

c. $2^? \div 2^3 = 2^{11}$

$2^{14} \div 2^3 = 2^{11}$

d. $\frac{10^{16}}{10^?} = 10^4$

$\frac{10^{16}}{10^{12}} = 10^4$

*e. $\frac{r^9}{r^?} = r^6$

$\frac{r^9}{r^3} = r^6$

*f. $\frac{f^8 f^3}{f^? y^?} = \frac{f^6}{y^2}$

$\frac{f^8 f^3}{f^5 y^2} = \frac{f^6}{y^2}$

6. Find the missing value in the following equations:

a. $(2^3)^4 = 2^?$

$(2^3)^4 = 2^{12}$

b. $(5^?)^6 = 5^{18}$

$(5^3)^6 = 5^{18}$

c. $(7^2)^? = 7^{10}$

$(7^2)^5 = 7^{10}$

*d. $(3^? \times 4^?)^2 = 3^4 \times 4^6$

$(3^2 \times 4^3)^2 = 3^4 \times 4^6$

*7. You will need to combine the rules you've learned to simplify these expressions.

a. $\dfrac{5^2 \times 5^8}{5^3}$

5^7

b. $(-4)^8 \times 4^4$

4^{12}

c. $\left(\dfrac{w^5 \times w^6}{w^8}\right)^3$

w^9